Final Exam Mathematical Physics, Prof. G. Palasantzas

- Date 23-06-2017
- Total number of points 100
- 10 points free for coming to the final exam
- Justify your answers for all problems

11-06-2017

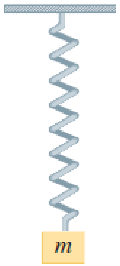
Problem 1 (10 points)

Prove that the series $\sum_{n=0}^{\infty} \frac{\sin^2(nx)}{2+5^n}$ is convergent

Problem 2 (10 points)

Find the range of convergence of the series $\sum_{n=1}^{\infty} \frac{\mathrm{n}}{\mathrm{b}^{\mathrm{n}}}(\mathrm{x}-\mathrm{c})^{\mathrm{n}} \quad (\mathrm{b}>0)$

Problem 3 (20 points)



Suppose a mass m is attached to a spring with spring constant k and let $k = m\omega_o^2$. If an external force $F(t) = F_o \cos(\omega t)$ is applied, then the equation of motion for non-zero dissipation ($c>0$) has the form:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \quad (1)$$

Using the method of undermined coefficients to prove that a particular solution of Eq. (1) $(c^2 - 4mk < 0)$ is given by:

$$x_p(t) = \left\{ \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \right\} \cos(\omega t + \delta)$$

$Tip: x(t) = c_1\cos(\omega t) + c_2\sin(\omega t) = A\cos(\omega t + \delta)$ with $\mathrm{A} = \sqrt{c_2^2 + c_1^2}$ and $\tan\delta = -c_2/c_1$

Problem 4 (20 points) Assume a function *f(x)* to have the Fourier transform: $F(k)=\int_{-\infty}^{+\infty} f(x)e^{-i2\pi kx}dx$

(a: 10 points) Calculate the Fourier Transform *F(k)* of the function: $f(x)=\begin{cases} e^{-ax} & \text{for x} \geq 0 \\ 0 & \text{for } x<0 \end{cases}$

(b: 10 points) For *F(k)* from (a) calculate $\lim_{a\to 0} F(k)$ by taking into account the definition of the Dirac delta function, $\delta(w)=(1/\pi)\lim_{a\to 0}\left(a/[w^2+a^2]\right)$, and $\delta(cw)=\delta(w)/|c|$.

Problem 5 (15 points) Assume a function *f(x)* to have Fourier Transform: $F(k)=\int_{-\infty}^{+\infty} f(x)e^{-i2\pi kx}dx$

Consider also the Fourier Transform definition of the Dirac Delta function: $\delta(k)=\int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$

Derive the Fourier Transform of : (a: 5 points) $f(x)=\cos[4\pi k_o x]$, (b: 10 points) $f(x)=\cos^2[4\pi k_o x]$

$Tip:\ \cos(x)=(e^{ix}+e^{-ix})/2$

Problem 6 (15 points)

Find the sine Fourier series solution to the differential equation $\frac{d^2y}{dx^2}+ky=f(x)$ with *k* an integer, *f(x)* an odd function [*f(x)=-f(-x)*], and the boundary conditions *y(0)=y(L)=0*

Tip: The Fourier sine expansion has the form $Y(x)=\sum_{n=1}^{\infty} b_n \sin(n\pi x/L),\ b_n=(2/L)\int_0^L Y(x)\sin(n\pi x/L)dx,\ \text{for x} \in [0,\text{L}]$

Problem 1

We have $\left|\frac{\sin^2(nx)}{2+5^n}\right| \le \frac{1}{2+5^n} < \frac{1}{5^n}$ 5 points

$\sum_{n=0}^{\infty} \frac{1}{5^n}$ Is a convergent geometric series with x=1/5<1.

Therefore, using the comparison test in comparison to the geometric series, our series is absolute convergent and thus $\sum_{n=0}^{\infty} \frac{\sin^2(nx)}{2+5^n}$ is convergent

5 points

Problem 2

$a_n = \frac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}} \cdot \frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$ 5 points

By the Ratio Test, the series converges when $\frac{|x-a|}{b} < 1 \iff |x-a| < b$ [so $R = b$] $\iff -b < x-a < b \iff a-b < x < a+b$. When $|x-a| = b$, $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

Range of convergence is (a-b, a+b) 5 points

Problem 3

In the following replace ω_0 with ω and you have the solution:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0 \cos\omega_0 t \quad (1)$$

$$X_p(t) = A(\omega_0)\cos(\omega_0 t) + B(\omega_0)\sin(\omega_0 t)$$

Substitute in (1) $\Rightarrow$

$$m(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + (-AC\omega_0\sin\omega_0 t + BC\omega_0\cos\omega_0 t) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0\cos\omega_0 t \Rightarrow$$

$$[(k-m\omega_0^2)A + CB\omega_0]\cos\omega_0 t + [(k-m\omega_0^2)B - CA\omega_0]\sin\omega_0 t = F_0\cos\omega_0 t \quad (2)$$

5 points

$(2) \Rightarrow$

$$(k-m\omega_0^2)A + CB\omega_0 = F_0 \quad (3)$$

$$(k-m\omega_0^2)B - CA\omega_0 = 0 \quad (4)$$

$(4) \Rightarrow$

$$B = A\frac{C\omega_0}{k-m\omega_0^2} \quad (5)$$

Substitute (5) into (3) $\Rightarrow$

$$A = \frac{F_0(k-m\omega_0^2)}{(k-m\omega_0^2)^2 + (C\omega_0)^2} \quad (6)$$

$$\Rightarrow B = \frac{F_0 C\omega_0}{(k-m\omega_0^2)^2 + (C\omega_0)^2} \quad (7)$$

10 points

The particular solution can be written as $X_p(t) = \tilde{A}(\omega_0)\cos(\omega_0 t + \delta)$

with

$$\tilde{A}(\omega) = \sqrt{A^2+B^2} = \frac{F_0}{\sqrt{(k-m\omega_0^2)^2 + (C\omega_0)^2}}$$

5 points

$$\tan\delta = -\frac{B}{A} = -\frac{C\omega_0}{k-m\omega_0^2} \Rightarrow$$

$$\delta = \tan^{-1}\left\{\frac{C\omega_0}{m\omega_0^2 - k}\right\}$$

Problem 4

(a)

$$\mathcal{F}[f(x)] = \int_0^{\infty} e^{-\alpha x} e^{-j2\pi kx} dx$$

$$\mathcal{F} = \int_0^{\infty} e^{-(\alpha+j2\pi k)x} dx = -\frac{1}{\alpha + j2\pi k}\left[e^{-(\alpha+j2\pi k)x}\Big|_{x=0}^{\infty} \right]$$ 5 points

$$\boxed{\mathcal{F} = +\frac{1}{\alpha + j2\pi k}}$$ 5 points

(b) $$\delta(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2}$$

Separate the real and imaginary parts of the Fourier transform from (a)

$$\mathcal{F} = \frac{\alpha - j2\pi k}{\alpha^2 + (2\pi k)^2} = \frac{\alpha}{\alpha^2 + (2\pi k)^2} - j\frac{2\pi k}{\alpha^2+(2\pi k)^2}$$ 5 points

$$\lim_{\alpha \to 0} \mathcal{F} = \underbrace{\lim_{\alpha \to 0} \frac{\alpha}{\alpha^2 + (2\pi k)^2}}_{\pi\,\delta(2\pi k)} - \frac{j}{2\pi k}$$

$$\lim_{\alpha \to 0} \mathcal{F} = \pi\,\delta(2\pi k) - \frac{j}{2\pi k}$$

$$\delta(2\pi k) = \frac{1}{2\pi}\delta(k)$$

5 points

$$\boxed{\mathcal{F}\left\{\lim_{\alpha \to 0} f(x)\right\} = \frac{1}{2}\left[\delta(k) - \frac{j}{\pi k}\right]}$$

Problem 5

(a) Here you can substitute directly to calculate the Fourier transform, so you have

$$F(k)=\frac{1}{2}[\int_{-\infty}^{+\infty} e^{-i2\pi(k-2k_o)x}dx+\int_{-\infty}^{+\infty} e^{-i2\pi(k+2k_o)x}dx]=\frac{1}{2}\{\delta(k-2k_o)+\delta(k+2k_o)\}$$ 5 points

(b) Here you can substitute directly to calculate the Fourier transform, so you have

$$F(k)=\frac{1}{4}\underbrace{[\int_{-\infty}^{+\infty} e^{-i2\pi(k-4k_o)x}dx+\int_{-\infty}^{+\infty} e^{-i2\pi(k+4k_o)x}dx+2\int_{-\infty}^{+\infty} e^{-i2\pi kx}dx]}_{\text{5 points}}=\underbrace{\frac{1}{4}\{\delta(k-4k_o)+\delta(k+4k_o)+2\delta(k)\}}_{\text{5 points}}$$

Problem 6 In the following replace **m with k** and you have the solution

Find the solution* of the differential equation

$y'' + my = f(x)$, m is integer, $f(x) = -f(-x)$ [odd function]

Under the conditions $y(0) = y(x=L) = 0$

* Sine solution.

$$y(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad (1)$$

Take $f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$, $C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$y''(x) = \sum_{n=1}^{\infty}\left(-\left(\frac{n\pi}{L}\right)^2\right) b_n \sin\left(\frac{n\pi x}{L}\right) \quad (2)$$

substitution into $y'' + my = f(x)$ gives

$$\sum_{n=1}^{\infty}\left[-b_n\left(\frac{n\pi}{L}\right)^2\right]\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} m b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$$ 5 points

$$\sum_{n=1}^{\infty}\left\{b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n\right\}\sin\left(\frac{n\pi x}{L}\right) = 0 \quad \forall x \in [0,L]$$

$$b_n\left[m - \left(\frac{n\pi}{L}\right)^2\right] - C_n = 0 \Rightarrow$$

$$b_n = \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2} \quad \left(m \neq \left(\frac{n\pi}{L}\right)^2\right)$$ 5 points

$$\boxed{y(x) = \sum_{n=1}^{\infty} \frac{C_n}{m - \left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)}$$ 5 points